FAST OBSERVERS FOR SPACECRAFT POINTING CONTROL

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ABSTRACT

This paper is concerned with designing fast stochastic observers for spacecraft pointing control. The motivation is to try to circumvent the long settling times associated with the optimal Kalman filter, which are often on the order of several hundred seconds in many typical spacecraft applications. For this purpose, the Karush-Kuhn-Tucker (KKT) necessary conditions are used to solve for an optimal constrained stochastic observer which minimizes the variance of the attitude estimate subject to the constraint that its poles lie to the left of a specified vertical line in the Laplace s-plane. A complete analytical solution is found for the typical case where the attitude estimator is comprised of three decoupled single-axis observers, each of second-order form. This results in a new fast observer design which can potentially improve operational efficiency over a broad class of spacecraft missions.

1 INTRODUCTION

Observers are typically flown on-board spacecraft to produce estimates of the state required by the pointing system for feedback control. If the observer has been optimized from a statistical point of view (i.e., it is a Kalman filter), these time constants are generally long (a few hundred seconds) for typical attitude estimators.

There is an unfortunate consequence of having a slow observer. Due to bias shifts which often occur in the position measurement (i.e., the star tracker) when changing from one attitude to the next, the observer's step response is seen explicitly in the pointing control response. This results in an undesirable drift error with a long settling time constant, occurring each time the telescope is repositioned. Because of this effect, it can take several hundred seconds for the control system to settle sufficiently before fine pointing can begin. This is somewhat disconcerting when one realizes that science exposures (for which the pointing is done in the first place) may only be on the order of 100 seconds or less. Hence, when flying the optimal Kalman filter (KF), much of the mission may be "wasted" waiting for the observer to settle. This waste of mission time is especially discouraging in missions (such as Infrared Telescopes) whose lifetime is defined by a finite amount of a depletable resource such as cryogen.

In order to overcome this difficulty, the present paper considers the problem of designing fast on-board stochastic observers. Rigorously speaking, these are constrained stochastic optimal observers which minimize the variance of the attitude estimate subject to the constraint that their poles lie to the left of a specified vertical line in the Laplace s-plane. Using the Karush-Kuhn-Tucker (KKT) necessary conditions, a globally optimal solution is found for the typical case where the attitude estimator is comprised of three decoupled single-axis observers, each of second-order form. This is the type of observer typically flown on space telescope missions such as IRAS, Hubble, etc.

Background on observers for attitude estimation is given in Section 2. The fast observer problem is then formulated in Section 3 as a nonlinear programming problem with nonconvex cost and constraints. To ensure a globally optimal solution, an approach is introduced to make the constraints linear. The main result of the paper, given in Section 4, is a globally optimal solution to the original nonlinear programming problem. Examples are given in Section 5 showing the optimal trade-off between observer speed and performance. Full technical details behind the solution are included in the Appendix.

2 BACKGROUND

In this paper, the spacecraft attitude estimator is assumed to be comprised of three decoupled single-axis observers. The analysis will be restricted to a single axis and be performed in continuous-time. This represents no loss in generality since each of the three axes can be designed separately with the same approach, and discrete-time implementations can be calculated using known transformations.

Consider the availability of a star tracker measurement of the form,

$$y = \theta + v \tag{2.1}$$

where θ is a small angle with respect to a local reference frame, and v is an additive white noise error source. Here, the star tracker is modeled as a continuous measurement along a single axis, and v is a continuous-time zero-mean white Gaussian noise source, where $E[v(t)v(t+\tau)] = r \cdot \delta(\tau)$.

Since star tracker measurements are always discrete, the quantity r is calculated as the power of an equivalent continuous-time noise process. For example, if a discrete-time star tracker measurement update is available every Δ seconds, r can be calculated as,

$$r = \Delta \sigma_{nea}^2 / N \tag{2.2}$$

where σ_{nea} is the noise equivalent angle in radians, 1-sigma per star, and N is the number of stars per update.

The quantity r is important because (as will be seen) it is the only aspect of the tracker design which effects the quality of the final attitude estimate. In fact, the inverse quantity r^{-1} has the interpretation of an "information rate" which can be used to compare the quality of any two tracker designs on equal footing.

In actual implementation, the local frame is defined by propagating the full attitude quaternion which ensures that the linearization and decoupling assumptions required for the present analysis are satisfied.

In addition, consider the availability of a single-axis gyro measurement of angular rate ω_m given by,

$$\omega_m = w - b - n_1 \tag{2.3}$$

$$\dot{b} = n_2 \tag{2.4}$$

Here, ω is the true angular rate, and n_1 and n_2 are independent, zero-mean white Gaussian noise sources, (denoted as the angle random walk (ARW) and bias instability, respectively), where $E[n_1(t)n_1(t+\tau)] = q_1 \cdot \delta(\tau)$, and $E[n_2(t)n_2(t+\tau)] = q_2 \cdot \delta(\tau)$. The quantity b is the gyro bias which generally has to be estimated in the observer along with the angular position.

Since rate is the derivative of position, one has,

$$\dot{\theta} = \omega \tag{2.5}$$

In order to avoid the need for modeling torques on the spacecraft, the gyro measurement (2.3) is treated as an "exogenous input" which is substituted into (2.5) to give,

$$\dot{\theta} = \omega_m + b + n_1 \tag{2.6}$$

Collecting state equations (2.4)(2.6) and measurement equation (2.1), gives,

$$\dot{\theta} = b + \omega_m + n_1 \tag{2.7}$$

$$\dot{b} = n_2 \tag{2.8}$$

$$y = \theta + v \tag{2.9}$$

By defining $u \triangleq \omega_m$ and $x \triangleq [\theta, b]^T$, equations (2.7)-(2.9) can be written in standard state-space form as,

$$\dot{x} = Ax + Bu + w \tag{2.10}$$

$$y = Cx + v \tag{2.11}$$

$$A \triangleq \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad B \triangleq \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{2.12}$$

$$C \triangleq \begin{bmatrix} 1 & 0 \end{bmatrix}; \quad w \triangleq \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \tag{2.13}$$

$$E[w(t)w^T(t+\tau)] = Q \cdot \delta(\tau); \quad Q = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} \tag{2.14}$$

$$E[v(t)v(t+\tau)] = r \cdot \delta(\tau) \tag{2.15}$$

A full-order observer for state space model (2.10)(2.11) can be written in the following standard form [3],

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - \hat{y}) \tag{2.16}$$

$$\hat{y} = C\hat{x} \tag{2.17}$$

where the gain matrix K has the form,

$$K = [k_1, k_2]^T \in R^2 \tag{2.18}$$

Let the state error be defined as,

$$e \triangleq x - \hat{x} \tag{2.19}$$

with associated covariance,

$$P \triangleq E[ee^T] = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \tag{2.20}$$

Then it is known that the covariance P propagates as [3],

$$\dot{P} = P(A - KC)^T + (A - KC)P + Q + r \cdot KK^T \tag{2.21}$$

Letting $\dot{P} = 0$ in (2.21), the resulting algebraic equations can be solved in closed form to give the steady-state covariance as,

$$p_{11} = \frac{r(k_2^2 + k_1^2 k_2) + q_1 k_2 + q_2}{2k_1 k_2} \tag{2.22}$$

$$p_{12} = \frac{r k_2^2 + q_2}{2k_2} \tag{2.23}$$

$$p_{22} = \frac{r(k_2^3 + k_1^2 k_2^2) + q_1 k_2^2 + q_2(k_2 + k_1^2)}{2k_1 k_2} \tag{2.24}$$

These expressions are useful because they show the explicit dependence of the estimation variances on the observer gains k_1 and k_2.

The characteristic polynomial of the observer (2.16)(2.17) can be calculated as,

$$\begin{aligned}\det(sI-(A-KC)) &= \det\begin{bmatrix} s+k_1 & -1 \\ k_2 & s \end{bmatrix} \\ &= s^2+k_1s+k_2 \end{aligned} \tag{2.25}$$

The observer poles are calculated as roots of (2.25), to give,

$$s = \frac{-k_1 \pm \sqrt{k_1^2-4k_2}}{2} \tag{2.26}$$

2.1 Optimal Stochastic Observer

The main focus of this paper will be on minimizing the variance of the angular position error p_{11} in (2.22), with respect to the observer gains k_1, k_2. To emphasize this dependence, we define the cost function,

$$J(k_1,k_2) \triangleq p_{11} = \frac{r(k_2^2+k_1^2k_2)+q_1k_2+q_2}{2k_1k_2} \tag{2.27}$$

The optimal stochastic observer (i.e., a Wiener filter) can be found by minimizing the variance (2.27) with respect to the observer gains k_1, k_2, i.e.,

$$\min_{k_1,k_2} J(k_1,k_2) \tag{2.28}$$

Strictly speaking, the gains should be constrained to be positive to ensure stability of the observer. However, this will not be necessary since it turns out that k_1 and k_2 satisfying the unconstrained problem (2.28) are always positive. Taking the derivatives of J and setting them equal to zero gives a set of algebraic equations,

$$\frac{\partial J}{\partial k_1} = 0 \tag{2.29}$$

$$\frac{\partial J}{\partial k_2} = 0 \tag{2.30}$$

These two algebraic equations can be solved simultaneously to give the optimal observer as,

Optimal Stochastic Observer

$$k_1^o = \left[2\cdot(\frac{q_2}{r})^{\frac{1}{2}} + \frac{q_1}{r}\right]^{\frac{1}{2}} \tag{2.31}$$

$$k_2^o = (\frac{q_2}{r})^{\frac{1}{2}} \tag{2.32}$$

The design (2.31)(2.32) is simply the steady-state Kalman filter and can be equivalently found by solving an algebraic Riccati equation. However, the approach taken here will allow us to introduce additional constraints on the observer poles.

The optimal gains (2.31)(2.32) can be substituted into (2.22)-(2.24) to give the optimal estimation variances,

$$p_{11} = \sqrt{r}\,(q_1 + 2\sqrt{q_2 r})^{\frac{1}{2}} \tag{2.33}$$

$$p_{12} = \sqrt{q_2 r} \tag{2.34}$$

$$p_{22} = q_2^{\frac{1}{2}}\,(q_1 + 2\sqrt{q_2 r})^{\frac{1}{2}} \tag{2.35}$$

Given gains (2.31)(2.32), the poles of the optimal observer can be computed from (2.26) as,

$$s = -\zeta\omega_n \pm w_n\sqrt{\zeta^2 - 1} \tag{2.36}$$

where,

$$\omega_n = k_2^{\frac{1}{2}} = (\frac{q_2}{r})^{\frac{1}{4}} \tag{2.37}$$

$$\zeta = \frac{1}{2}\left(2 + \frac{q_1}{\sqrt{q_2 r}}\right)^{\frac{1}{2}} \tag{2.38}$$

For convenience, the circle in the s-plane specified by $|s| = \omega_n$ will be denoted as the "Kalman circle".

It is instructive to consider the root locus as the parameter q_1 is increased relative to $2\sqrt{q_2 r}$. When $q_1 \leq 2\sqrt{q_2 r}$, the poles are complex. Specifically, it is seen that when $q_1 = 0$, the system starts out critically damped with $\zeta = 1/\sqrt{2}$. As q_1 is increased the damping factor ζ increases, and the complex poles move along the Kalman circle toward the real axis. They hit the real axis when $q_1 = 2\sqrt{q_2 r}$ to become repeated roots. When $q_1 > 2\sqrt{q_2 r}$, the poles become split-real, and go off in opposite directions – one towards the origin, and one towards $-\infty$.

In general, these poles may be too sluggish to support efficient on-orbit spacecraft operations. A particularly bad situation arises from the split-root case where one of the poles moves toward the origin as q_1 is increased relative to $2\sqrt{q_2 r}$. However, even in the best case when the optimal poles are complex and critically damped (i.e., when $q_1 = 0$), the observer may be sluggish due to the fact that ω_n is a low frequency (i.e., the optimal filter needs to smooth over a long data window to minimize the variance of the estimate).

2.2 Optimal Constrained Stochastic Observers

Instead of the unconstrained problem (2.28), consider the modified problem,

Fast Observer

$$\min_{k_1, k_2} \quad J(k_1,\ k_2) \tag{2.39}$$

subject to,

$$\text{Real}\{\text{poles of } s^2 + k_1 s + k_2\} \ \leq -\frac{1}{\tau} \tag{2.40}$$

This problem minimizes the estimation variance subject to the constraint that the poles of the observer are to the left of a vertical line in the Laplace s-plane at $-1/\tau$. This ensures that the longest *time constant* of the observer is shorter than τ.

REMARK 2.1 It is worth noting that problem (2.39)(2.40) is equivalent to the following Bilinear Matrix Inequality (BMI):

$$\min_{K,\ P>0,\ M>0} \quad Trace\{WP\} \tag{2.41}$$

$$P(A-KC)^T + (A-KC)P + Q + r\cdot KK^T = 0 \tag{2.42}$$

$$M(A-KC)^T + (A-KC)M + \frac{2}{\tau}\cdot M < 0 \tag{2.43}$$

where W is a diagonal weighting matrix.

The main result of the paper is a complete analytic solution to problem (2.39)(2.40).

3 FORMULATION

3.1 Pole Location Parametrization

Instead of using observer gains, it will be useful to reparametrize the problem in terms of pole locations. To this end, let the poles of the observer be denoted as p_1, p_2 where,

$$p_1 \ \triangleq \ -a_1 + jb \tag{3.1}$$

$$p_2 \ \triangleq \ -a_2 - jb; \ \ (j \triangleq \sqrt{-1}) \tag{3.2}$$

and such that the quantities a_1, a_2, b are constrained according to,

$$b(a_1 - a_2) = 0 \tag{3.3}$$

This constraint is motivated by the observation that if $b \neq 0$ the poles form a complex conjugate pair in which case $a_1 = a_2$. However, if $b = 0$ the poles are real and a_1 does not necessarily equal a_2.

Using (3.1)(3.2) and (3.3), one can compute,

$$\begin{aligned} s^2 + k_1 s + k_2 &= (s - p_1)(s - p_2) = s^2 - (p_1 + p_2)s + p_1 p_2 \\ &= s^2 + (a_1 + a_2)s + a_1 a_2 + b^2 \end{aligned} \tag{3.4}$$

or equivalently,

$$k_1 = a_1 + a_2 \tag{3.5}$$

$$k_2 = a_1 a_2 + b^2 \tag{3.6}$$

These expressions can be combined to write the constrained problem (2.39) as the following single nonlinear programming problem,

$$\min_{a_1, a_2, b} \quad J(a_1 + a_2,\ a_1 a_2 + b^2) \tag{3.7}$$

subject to,

$$b(a_2 - a_1) = 0 \tag{3.8}$$

$$a_1 \geq \frac{1}{\tau} \tag{3.9}$$

$$a_2 \geq \frac{1}{\tau} \tag{3.10}$$

$$b \geq 0 \tag{3.11}$$

where the cost function $J(k_1, k_2)$ is defined in (2.27). The restriction $b \geq 0$ is made without loss of generality, to avoid redundant solutions from complex pole symmetry about the real axis. The equality constraint (3.8) can be satisfied in only one of two possible ways: Either $b = 0$, in which case the poles are both real and a_1 and a_2 separately satisfy the half-plane constraints (3.9)(3.10); or $b > 0$, in which case $a_1 = a_2$, and the poles are complex conjugate with the same real part.

Unfortunately, the constraint region (3.8)-(3.11) is not convex, and does not even have interior points (i.e., the constraint set lies on a two-dimensional manifold in a three-dimensional space). For this type of constraint region, it is well-known that the Karush-Kuhn-Tucker (KKT) conditions may fail to be necessary conditions for a local minima [2][5]. Instead, the problem will be reformulated next to ensure that the KKT conditions are necessary.

3.2 Linear Constraint Reformulation

Constraints are reformulated to be linear by considering the real and complex root cases separately in problem (3.7).

Real Root Case

If the roots are real, then $b = 0$ and the gains are given by $k_1 = a_1 + a_2$ and $k_2 = a_1 a_2$. Hence, the cost $J(k_1, k_2)$ becomes a function of only the two parameters a_1 and a_2. The resulting real-root optimization problem can be stated as,

$$\min_{a_1, a_2} \quad J(a_1 + a_2, \ a_1 a_2) \tag{3.12}$$

subject to,

$$a_1 \geq \frac{1}{\tau} \tag{3.13}$$

$$a_2 \geq \frac{1}{\tau} \tag{3.14}$$

It is noted that constraints (3.13)(3.14) are linear.

Complex Root Case

If the roots are complex, then the real parts are the same and one may invoke the notation $a \triangleq a_1 = a_2$. In this case the gains are given by $k_1 = 2a$ and $k_2 = a^2 + b^2$. The cost $J(k_1, k_2)$ becomes a function of only the two parameters a and b. The resulting complex-root optimization problem can be stated as,

$$\min_{a,b} \quad J(2a, \ a^2 + b^2) \tag{3.15}$$

subject to,

$$a \geq \frac{1}{\tau} \tag{3.16}$$

It is noted that the constraint (3.16) is linear.

For simplicity, the limiting case of a repeated real-root solution has been permitted to exist in both the above cases.

Since the roots of the observer must be either real or complex (there are no other choices) *the global optimal of the original problem (3.7) must be one of the two global optimals arising from considering each problem (3.12) and (3.15) separately.* Furthermore, since constraint sets in both problems are now linear, it is known that the KKT conditions (cf., (A.4)-(A.7) in Appendix A) are necessarily satisfied by the global constrained minimum of each problem when considered separately [2][5].

This leads to the following strategy for finding the global constrained optimal to the original problem:

- Find all of the points which satisfy the KKT conditions for the real root case (3.12), and then do the same for the complex root case (3.15). Combine these points into a single list, which then represents a complete set of candidate solutions to the original problem.

- Evaluate the cost of each candidate solution on the list. The one that minimizes the cost is the global constrained optimal solution to the original constrained problem (3.7).

It will be seen that the "list" of candidate solutions produced in the first step is always finite, so that the overall procedure is guaranteed to be finite. The list of candidate solutions satisfying the KKT conditions for the real and complex cases separately, is determined systematically in Appendix B.

3.3 Normalized Variables

It will be convenient to express the final solution in terms of normalized variables:

$$\mu \triangleq \frac{q_1}{r} \cdot \tau^2; \quad \nu \triangleq \frac{q_2}{r} \cdot \tau^4 \tag{3.17}$$

$$x_1 \triangleq a_1\tau; \quad x_2 \triangleq a_2\tau; \quad x_3 \triangleq b\tau \tag{3.18}$$

The observer gains (3.5)(3.6) can be written in terms of these normalized variables as,

$$k_1 = (x_1 + x_2)/\tau \tag{3.19}$$

$$k_2 = (x_1x_2 + x_3^2)/\tau^2 \tag{3.20}$$

Substituting gains (3.19)(3.20) into the cost function $J(k_1, k_2)$ in problem (3.7) yields upon rearranging,

$$J(x_1, x_2, x_3) = \frac{r}{2\tau} \cdot \bar{J}(x_1, x_2, x_3) \tag{3.21}$$

where,

$$\bar{J} = \left[x_1 + x_2 + (x_1 + x_2)^{-1}\left(x_1x_2 + x_3^2 + \mu + \nu\left(x_1x_2 + x_3^2\right)^{-1}\right)\right] \tag{3.22}$$

Letting $x \triangleq [x_1, x_2, x_3]$, the problem (3.7) can be written equivalently as,

$$\min_x \bar{J}(x) \tag{3.23}$$

subject to,

$$x_3(x_1 - x_2) = 0 \tag{3.24}$$

$$1 - x_1 \leq 0 \tag{3.25}$$

$$1 - x_2 \leq 0 \tag{3.26}$$

$$- x_3 \leq 0 \tag{3.27}$$

Using the second-derivative test given in Rockafellar [6] (Theorem 4.5, pp. 27), the cost function $\bar{J}$ can be shown to be *non-convex*. The quantity $\bar{J}$ has been used as the cost in the minimization (3.23) instead of J. This represents no loss of generality since they are related by a constant scale factor (3.21).

For a given time constant τ, any candidate solutions x_1, x_2, x_3 to the minimization problem (3.23), can be converted to the associated pole locations using (3.18).

4 CONSTRAINED PROBLEM SOLUTION

4.1 Main Result

The main result of this paper is an analytic globally optimal solution to the nonlinear programming problem (3.23). To this end, a finite list of candidate solutions are generated using the following rules.

Case R-I(i) *(Optimal real-pole Kalman filter solution)*

$$[x_1, x_2, x_3] = \left[\left(\frac{\mu + \sqrt{\mu^2 - 4\nu}}{2}\right)^{\frac{1}{2}}, \left(\frac{\mu - \sqrt{\mu^2 - 4\nu}}{2}\right)^{\frac{1}{2}}, 0\right] \tag{4.1}$$

is a candidate solution if x_1 and x_2 are real, and the inequalities $x_1 \geq 1$, $x_2 \geq 1$ hold.

Case R-I(ii) *(Repeated real poles)*

If the following inequality holds,

$$\left[\frac{\mu + \sqrt{\mu^2 + 60\nu}}{10}\right]^{\frac{1}{2}} \geq 1 \tag{4.2}$$

then a candidate solution is given by,

$$[x_1, x_2, x_3] = \left[\left(\frac{\mu + \sqrt{\mu^2 + 60\nu}}{10}\right)^{\frac{1}{2}}, \left(\frac{\mu + \sqrt{\mu^2 + 60\nu}}{10}\right)^{\frac{1}{2}}, 0\right] \tag{4.3}$$

Case R-II *(Split real poles, one at $s = -1/\tau$)*

Compute the roots λ_i, $i = 1, \ldots, 4$ of the following fourth order polynomial,

$$p(x) = x^4 + 2x^3 + (2 - \mu)x^2 - 2\nu x - \nu = 0 \tag{4.4}$$

For each λ_i which is real and satisfies $\lambda_i \geq 1$, a candidate solution is given by,

$$[x_1, x_2, x_3] = [\lambda_i, 1, 0] \tag{4.5}$$

(Note, at most four candidate solutions can be generated from this single case).

Case R-IV *(Repeated real poles at $s = -1/\tau$)*

A candidate solution is given by,

$$[x_1, x_2, x_3] = [1, 1, 0] \tag{4.6}$$

Case C-I *(Optimal complex-pole Kalman filter solution)*

If the following inequalities hold,

$$\frac{1}{2}\left[2\sqrt{\nu} + \mu\right]^{\frac{1}{2}} \geq 1 \tag{4.7}$$

$$\mu \leq 2\sqrt{\nu} \tag{4.8}$$

then a candidate solution is given by,

$$[x_1, x_2, x_3] = \left[\frac{1}{2}\left(2\sqrt{\nu} + \mu\right)^{\frac{1}{2}}, \frac{1}{2}\left(2\sqrt{\nu} + \mu\right)^{\frac{1}{2}}, \frac{1}{2}\left(2\sqrt{\nu} - \mu\right)^{\frac{1}{2}}\right] \tag{4.9}$$

Case C-II *(Complex poles on Kalman circle $|s| = \omega_n$ at Real(s) $= -1/\tau$)*

If $\sqrt{\nu} \geq 1$, then a candidate solution is given by,

$$[x_1, x_2, x_3] = [1, 1, (\sqrt{\nu} - 1)^{\frac{1}{2}}] \tag{4.10}$$

A sweep through the above rules produces a list of candidate solutions. By construction, the globally optimal constrained solution to the original problem (3.23) is guaranteed to be contained on this list. It is found by evaluating all of the candidate solutions, and keeping the one with minimum cost (i.e, minimum variance).

5 EXAMPLES

Fast observer designs will be applied to two examples. The first example is chosen such that the optimal Kalman filter has complex poles. The second example is chosen such that the optimal KF has real poles.

5.1 Example 1: Complex Pole KF

Consider the hardware parameters, $q_1 = 8.462 \times 10^{-18}$ $(rad^2/sec^2)/Hz$, $q_2 = 6.529 \times 10^{-21}$ $(rad^2/sec^4)/Hz$, $r = 1.653 \times 10^{-12}$ rad^2/Hz.

The optimal Kalman filter as computed from (2.31)-(2.32) has gains $k_1^o = .011438$, $k_2 = 6.2854 \times 10^{-5}$, with complex pole locations $-.00572 \pm .005491$ and a time constant of 175 seconds. Using results in Section 4, fast observers having reduced time constants of $\tau = 180, 140, 100, 60, 20$ are designed and their performance is summarized in Table 5.1. The constraint regions which are traversed are depicted in Figure 5.1 For physical meaningfulness, the cost is expressed as the square root $\sqrt{J}$ of the estimation variance, in units of arcseconds.

For this example, it is seen that (except for the $\tau = 800$ case) both poles of the constrained design always lie on the constraint boundary Real(s) $= -1/\tau$. For $\tau = 180$ (which is longer than the unconstrained optimal KF time constant of 172), the constrained and unconstrained optimals are the same (region C-I). As τ is decreased further, the poles follow the constraint boundary (region C-I) as complex conjugate pairs on the Kalman circle $|s| = \omega_n$. This continues until they become repeated-real poles at

$$\tau = \frac{1}{\omega_n} = (\frac{r}{q_2})^{\frac{1}{4}} = 126.14\ (sec) \tag{5.1}$$

and then follow the constraint boundary (region R-IV) as repeated roots on the real axis.

Fast Observers

τ	Case#	x_1	x_2	x_3	k_1	k_2	poles	$\sqrt{J}$(as)
180	C-I	1.0294	1.0294	.98831	.01144	6.285e-5	$-.00572 \pm .00549j$	.02836”
174.9	Opt KF	NA	NA	NA	.01144	6.285e-5	$-.00572 \pm .00549j$	.02836”
140	C-II	1	1	.4816	.01429	6.285e-5	$-.00714 \pm .00344j$	.02871”
100	R-IV	1	1	0	.02000	1.000e-4	$-.01,\ -.01$	.03094”
60	R-IV	1	1	0	.03333	2.778e-4	$-.0167,\ -.0167$	.03854”
20	R-IV	1	1	0	.10000	2.500e-3	$-.05,\ -.05$	.06631”

Table 5.1: Designs for the case of $q_1 = 8.462 \times 10^{-18}, q_2 = 6.529 \times 10^{-21}, r = 1.653 \times 10^{-12}$

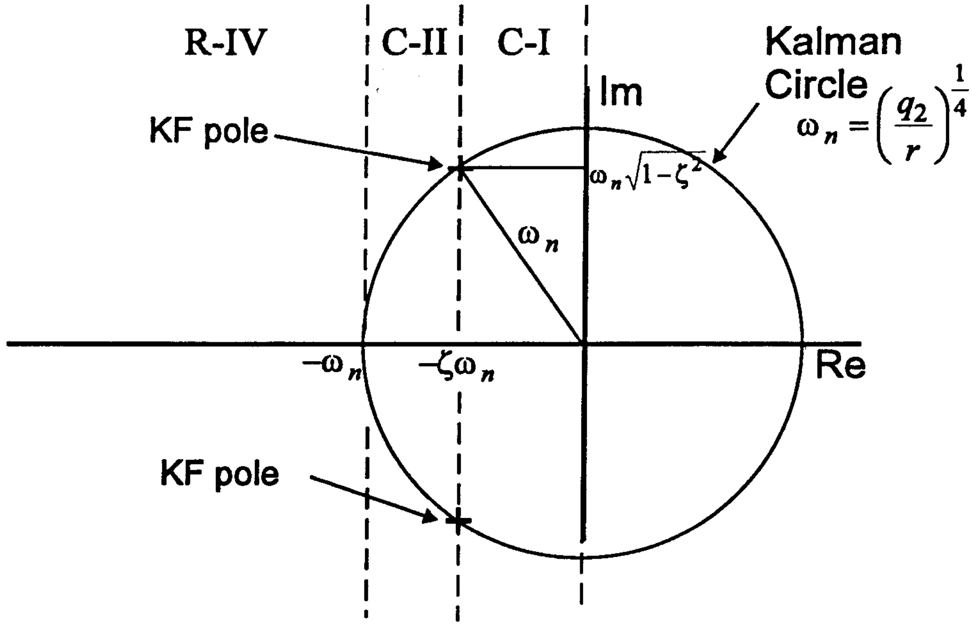

Figure 5.1: Constraint regions when detuning a complex-pole Kalman filter

5.2 Example 2: Real Pole KF

Consider the hardware parameters $q_1 = 3.385 \times 10^{-15}$ $(rad^2/sec^2)/Hz$, $q_2 = 6.529 \times 10^{-21}$ $(rad^2/sec^4)/Hz$, $r = 1.653 \times 10^{-12}$ rad^2/Hz.

The optimal Kalman filter has gains $k_1 = .046623$, $k_2 = 6.2854 \times 10^{-5}$, with real-split pole locations at $-.04523, -.001390$, and a time constant of 720 seconds. As discussed earlier, the KF is sluggish due to a single slow real pole, which always occurs under the split-root condition $q_1 >> 2\sqrt{q_2 r}$. To overcome this sluggishness, fast observers having reduced time constants of $\tau = 800, 180, 140, 100, 60, 20$ are designed and their performance is summarized in Table 5.2. The constraint regions which are traversed are depicted in Figure 5.2.

For $\tau = 800$ (which is longer than the unconstrained optimal KF time constant of 720), the constrained and unconstrained optimals are the same (region R-I(i)). As τ is decreased from 180 to 60, the pole closest to the origin moves left with the constraint boundary while the other pole (deeper in the LHP) moves right to meet it (region R-II). After they meet, both poles follow the constraint boundary as repeated roots on the real axis (region R-IV).

The break-point between regions R-II and R-IV occurs at the point where the two real poles meet. This can be found by setting $x_1 = x_2 = 1$ in the polynomial (A.30) associated with region R-II, to give ,

$$\mu + 3\nu = 5 \tag{5.2}$$

Substituting (3.17) into (5.2) and solving for τ as the only positive root gives the break

Fast Observers

τ	Case#	x_1	x_2	x_3	k_1	k_2	poles	$\sqrt{J}$(as)
800	R-I(i)	36.187	1.1116	0	.046623	6.2854e-5	$-.04523, -.001390$	.057255"
720	Opt KF	NA	NA	NA	.046623	6.2854e-5	$-.04523, -.001390$	.057255"
180	R-II	7.1605	1	0	.045336	2.2100e-4	-.039781, -.005556	.058028"
140	R-II	5.3061	1	0	.045043	2.7072e-4	-.037901, -.0071429	.058350"
100	R-II	3.4433	1	0	.044433	3.4433e-4	$-.034433, -.01000$	.058856"
60	R-II	1.5418	1	0	.042363	4.2828e-4	$-.025697, -.016667$	.059599"
20	R-IV	1	1	0	.01000	2.5000e-3	$-.05000, -.05000$	.071520"

Table 5.2: Designs for the case of $q_1 = 3.385 \times 10^{-15}, q_2 = 6.529 \times 10^{-21}, r = 1.653 \times 10^{-12}$

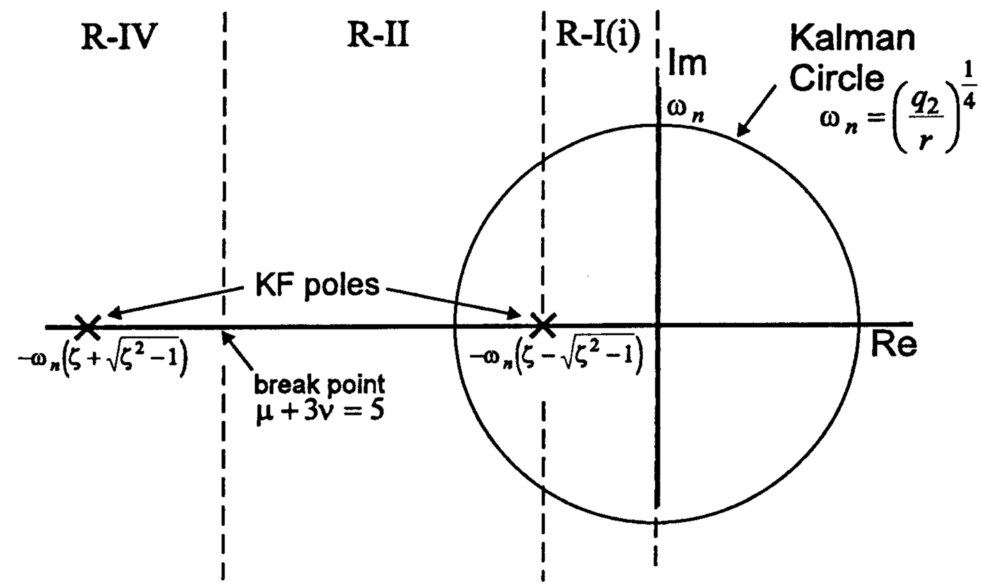

Figure 5.2: Constraint regions when detuning a real-pole Kalman filter

point as,

$$\tau = \left[\frac{-\frac{q_1}{r} + \sqrt{(\frac{q_1}{r})^2 + 60(\frac{q_2}{r})}}{6(\frac{q_2}{r})} \right]^{\frac{1}{2}} = 49.07 \ (sec) \tag{5.3}$$

Before this break point, the fast observer reduces the time constant of $\tau = 720$ seconds (associated with the optimal KF), by more than a factor of 10 to $\tau = 60$, with less than 5% degradation in the estimation error (i.e., from .057255 to .059599 arcseconds). After the break point, performance falls off somewhat quicker, with the $\tau = 20$ design providing an estimation error of .071520 arcseconds.

6 CONCLUSIONS

A complete solution is found for designing constrained stochastic observers of second-order form, which minimize the variance of the attitude estimate subject to the constraint their poles lie to the left of a specified vertical line in the Laplace s-plane. Such fast observers can potentially improve operational efficiency across a wide class of spacecraft missions. These results also support the reconfigurable control approach introduced in [1] which improves performance by switching among a bank of detuned observers.

Examples are given showing the optimal trade-off between fast observer speed and performance. While the approach may be useful across a wide range of applications, the advantages are particularly pronounced in problems where the optimal KF has split real roots. An example of this form was given where the time constant of the optimal Kalman filter is reduced by factor of 10, with less than a 5% degradation in estimation error.

A APPENDIX: Nonlinear Programming Solution

Consider a general nonlinear programming problem of the form,

$$\min_{x \in R^n} \; J(x) \tag{A.1}$$

subject to,

$$g_i(x) \leq 0, \quad i = 1, ..., m \tag{A.2}$$

Assume that *the cost function $J(x)$ is differentiable, but not necessarily convex.* Assume further that *the constraints $g(x) = [g_1, ..., g_m]^T$ are linear,* i.e.,

$$g(x) = Cx + d, \quad C \in R^{m \times n}, \; d \in R^m \tag{A.3}$$

Then it is known (Karlin [4], p. 203) that under these conditions, any local minimum x^o of problem (A.1) (and the global minimum in particular) must necessarily satisfy the following KT conditions:

$$\nabla J(x^o) = -\sum_{i=1}^{m} u_i^o \nabla g_i(x^o) \tag{A.4}$$

$$g_i(x^o) \leq 0, \tag{A.5}$$

$$u_i^o \geq 0, \tag{A.6}$$

$$u_i^o g_i(x^o) = 0; \quad i = 1, ..., m \tag{A.7}$$

The importance of having necessary conditions cannot be overstated. It implies that by enumerating all possible solutions to the KT conditions, one is guaranteed to find the global constrained optimal among them.

A.1 Real Root Case $x_3 = 0$

Letting $x_3 = 0$ in problem (3.23), restricts its solution to the real-root case. With this restriction, the problem can be written equivalently as a nonlinear program in the form (A.1)(A.2) with the following choices,

$$\min_x \; J^R(x) \tag{A.8}$$

$$x \triangleq [x_1, x_2]^T \tag{A.9}$$

$$J^R(x) \triangleq \left[x_1 + x_2 + (x_1 + x_2)^{-1}\left(x_1 x_2 + \mu + \nu(x_1 x_2)^{-1}\right)\right] \tag{A.10}$$

$$g^R(x) \triangleq \begin{bmatrix} 1 - x_1 \\ 1 - x_2 \end{bmatrix} \leq 0 \tag{A.11}$$

Applying the KKT condition (A.4), gives,

$$\nabla J^R(x) = \begin{bmatrix} \frac{\partial J^R}{\partial x_1} \\ \frac{\partial J^R}{\partial x_2} \end{bmatrix} \tag{A.12}$$

$$= \begin{bmatrix} 1 - (x_1 + x_2)^{-2}(x_1 x_2 + \mu + \nu(x_1 x_2)^{-1}) + (x_1 + x_2)^{-1}\left(x_2 - \nu x_1^{-2} x_2^{-1}\right) \\ 1 - (x_1 + x_2)^{-2}(x_1 x_2 + \mu + \nu(x_1 x_2)^{-1}) + (x_1 + x_2)^{-1}\left(x_1 - \nu x_1^{-1} x_2^{-2}\right) \end{bmatrix}$$

$$= -u_1^o \nabla g_1^R(x) - u_2^o \nabla g_2^R(x) = \begin{bmatrix} u_1^o \\ u_2^o \end{bmatrix} \tag{A.13}$$

Now consider the four possible cases determining which constraints are active:

Case R-I $[u_1^o = 0, u_2^o = 0]$

Case R-II $[u_1^o = 0, u_2^o > 0]$

Case R-III $[u_1^o > 0, u_2^o = 0]$

Case R-IV $[u_1^o > 0, u_2^o > 0]$

Case R-I $[u_1^o = 0, u_2^o = 0]$

The choice $[u_1^o = 0, u_2^o = 0]$ implies from (A.13) that

$$\nabla J^R(x) = 0 \tag{A.14}$$

Hence, one can equate,

$$\frac{\partial J^R}{\partial x_1} = \frac{\partial J^R}{\partial x_2} \tag{A.15}$$

and cancel terms to give the expression,

$$(x_1 + x_2)^{-1} \left[x_2 - \frac{\nu}{x_1^2 x_2} \right] = (x_1 + x_2)^{-1} \left[x_1 - \frac{\nu}{x_1 x_2^2} \right] \tag{A.16}$$

Now $x_1 \geq 1$ and $x_2 \geq 1$ imply that $x_1 + x_2 > 0$, so that equation (A.16) is equivalent to,

$$\left[x_2 - \frac{\nu}{x_1^2 x_2} \right] = \left[x_1 - \frac{\nu}{x_1 x_2^2} \right] \tag{A.17}$$

or upon rearranging,

$$(x_1^2 x_2^2 - \nu) x_2 = (x_1^2 x_2^2 - \nu) x_1 \tag{A.18}$$

From (A.18), one can conclude only two possibilities:

Case R-I(i) $x_1^2 x_2^2 = \nu$

Case R-I(ii) $x_1^2 x_2^2 \neq \nu$ which implies $x_1 = x_2$

These sub-cases are treated separately below.

Case R-I(i)

Substituting

$$x_1^2 x_2^2 = \nu \tag{A.19}$$

into the expression, $\frac{\partial J^R}{\partial x_1} = 0$ of (A.13), gives upon rearranging,

$$x_1^2 + \frac{\nu}{x_1^2} - \mu = 0 \tag{A.20}$$

Since $x_1 \geq 1$ is non-zero, one can multiply both sides of (A.20) by x_1^2 to give,

$$x_1^4 - \mu x_1^2 + \nu = 0 \tag{A.21}$$

Solving for roots of the fourth order polynomial (A.21) explicitly gives,

$$x_1 = + \left[\frac{\mu \pm \sqrt{\mu^2 - 4\nu}}{2} \right]^{\frac{1}{2}} \tag{A.22}$$

where only the positive root is considered because $x_1 \geq 1$ ensures nonnegative values of x_1. Knowing x_1, one can solve for x_2 from (A.19) to give,

$$x_2 = \frac{\sqrt{\nu}}{x_1} \tag{A.23}$$

Here, the negative root $-\sqrt{\nu}$ is not considered since the left hand side must always be nonnegative. Without loss of generality, one can always choose the positive root on the radical in (A.22), because the expression for x_2 in (A.23) gives the corresponding negative root, i.e.,

$$x_1 = +\left[\frac{\mu + \sqrt{\mu^2 - 4\nu}}{2}\right]^{\frac{1}{2}}; \quad x_2 = +\left[\frac{\mu - \sqrt{\mu^2 - 4\nu}}{2}\right]^{\frac{1}{2}} \tag{A.24}$$

Equation (A.24) provides the desired expressions for x_1 and x_2.

Case R-I(ii)

In this case, $x_1^2 x_2^2 \neq \nu$ holds in (A.18) which implies that

$$x_1 = x_2 \tag{A.25}$$

Substituting (A.25) into the expression $\frac{\partial J^R}{\partial x_1} = 0$ of (A.13) gives upon rearranging,

$$5x_1^4 - \mu x_1^2 - 3\nu = 0 \tag{A.26}$$

Solving the fourth order polynomial (A.26) explicitly gives,

$$x_1 = \pm\left[\frac{\frac{\mu}{5} \pm \sqrt{\frac{\mu^2}{25} + \frac{12}{5}\nu}}{2}\right]^{\frac{1}{2}} \tag{A.27}$$

Given the constraint $x_1 \geq 1$ and the fact that x_1 must be real-valued, only one of the four solutions in (A.27) needs to be considered, specifically,

$$x_1 = +\left[\frac{\frac{\mu}{5} + \sqrt{\frac{\mu^2}{25} + \frac{12}{5}\nu}}{2}\right]^{\frac{1}{2}} = \left[\frac{\mu + \sqrt{\mu^2 + 60\nu}}{10}\right]^{\frac{1}{2}} \tag{A.28}$$

Equations (A.28) and (A.25) are the desired expressions for x_1 and x_2.

Case R-II $[u_1^o = 0, u_2^o > 0]$

The choice $u_1^o = 0, u_2^o > 0$ implies that $g_2^R(x) = 1 - x_2 = 0$, i.e.,

$$x_2 = 1 \tag{A.29}$$

Substituting $x_2 = 1$ into the expression $\frac{\partial J^R}{\partial x_1} = 0$ of (A.13) gives upon rearranging,

$$x_1^4 + 2x_1^3 + (2-\mu)x_1^2 - 2\nu x - \nu = 0 \tag{A.30}$$

Roots of (A.30), denoted by $x_1 = \lambda_i$, $i = 1, \ldots, 4$, which are real and satisfy $x_1 \geq 1$ are candidate choices for x_1.

Case R-III $[u_1^o > 0, u_2^o = 0]$

Case R-III is omitted because it gives identical redundant solutions to Case R-II, (with the labels of x_1 and x_2 reversed). To see this, make the replacements,

$$x_1 \leftrightarrow x_2 \tag{A.31}$$

$$u_1^o \leftrightarrow u_2^o \tag{A.32}$$

in the KKT condition (A.13), and note that the condition is left unchanged.

Case R-IV $[u_1^o > 0, u_2^o > 0]$

The choice $u_1^o = 0, u_2^o > 0$ implies that $g(x) = 0$, i.e., both state constraints are satisfied with equality,

$$x_1 = 1 \tag{A.33}$$

$$x_2 = 1 \tag{A.34}$$

A.2 Complex Root Case $x_2 = x_1$

Letting $x_2 = x_1$ in problem (3.23) restricts its solution to the complex-root case. With this restriction, the problem can be written equivalently as a nonlinear program in the form (A.1)(A.2) with the following choices,

$$\min_x \; J^C(x) \tag{A.35}$$

$$x \triangleq [x_1, x_3]^T \tag{A.36}$$

$$J^C(x) = \left[2x_1 + \frac{1}{2x_1}\left(x_1^2 + x_3^2 + \mu + \frac{\nu}{x_1^2 + x_3^2}\right)^{-1}\right] \tag{A.37}$$

$$g^C(x) = 1 - x_1 \le 0 \tag{A.38}$$

Applying the KKT conditions, gives,

$$\begin{aligned}
\nabla J^C(x) &= \begin{bmatrix} \frac{\partial J^C}{\partial x_1} \\ \frac{\partial J^C}{\partial x_3} \end{bmatrix} \\
&= \begin{bmatrix} 2 - \frac{x_1^{-2}}{2}\left(x_1^2 + x_3^2 + \mu + \frac{\nu}{x_1^2+x_3^2}\right) + \frac{x_1^{-1}}{2}\left(2x_1 - 2\nu x_1(x_1^2 + x_3^2)^{-2}\right) \\ \frac{1}{2x_1}\left(2x_3 - 2\nu x_3(x_1^2 + x_3^2)^{-2}\right) \end{bmatrix} \\
&= -u_1^o \nabla g_1^C(x) = \begin{bmatrix} u_1^o \\ 0 \end{bmatrix}
\end{aligned} \tag{A.39}$$

Now consider the two possible cases:

Case C-I $[u_1^o = 0]$

Case C-II $[u_1^o > 0]$

Case C-I $[u_1^o = 0]$

The choice $u_1^o = 0$ implies from (A.39) that

$$\nabla J^C(x) = 0 \tag{A.40}$$

This gives 2 equations in 2 unknowns. Specifically, from the bottom equation $\frac{\partial J^C}{\partial x_3} = 0$ of (A.39) one has (given that $x_1 \geq 1$),

$$2x_3 = 2\nu x_3(x_1^2 + x_3^2)^{-2} \tag{A.41}$$

which gives upon cancelling terms,

$$x_1^2 + x_3^2 = \sqrt{\nu} \tag{A.42}$$

Here, the negative root $-\sqrt{\nu}$ is not considered since the left hand side must always be nonnegative.

From the top equation $\frac{\partial J^C}{\partial x_1} = 0$ of (A.39) one has upon rearranging,

$$x_1 = +\left(\frac{2\sqrt{\nu} + \mu}{4}\right)^{\frac{1}{2}} \tag{A.43}$$

Here, the negative square root is not considered because x_1 must be non-negative due to constraint $x_1 \geq 1$. Substituting (A.43) into (A.42) and rearranging gives,

$$x_3 = +\left(\frac{2(\sqrt{\nu}) - \mu}{4}\right)^{\frac{1}{2}} \tag{A.44}$$

Here, the negative square root is not considered to avoid redundant identical solutions (i.e., x_3 is the imaginary part of one of a pair of complex conjugate poles). Equations (A.43) and (A.44) are the desired expressions for x_1 and x_3, respectively.

Case C-II $[u_1^o > 0]$

The choice $u_1^o > 0$ implies from the KT condition (A.7) that $g^C(x) = 1 - x_1 = 0$, i.e.,

$$x_1 = 1 \tag{A.45}$$

From the bottom equation $\frac{\partial J^C}{\partial x_3} = 0$ of (A.39) one has (upon cancelling terms),

$$x_1^2 + x_3^2 = \sqrt{\nu} \tag{A.46}$$

Here, the negative root $-\sqrt{\nu}$ is not considered since the left hand side must always be nonnegative. Substituting (A.45) into (A.46) gives upon rearranging,

$$x_3 = (\sqrt{\nu} - 1)^{\frac{1}{2}} \tag{A.47}$$

Here, the negative square root is not considered to avoid redundant identical solutions. Equations (A.45) and (A.47) are the desired expressions for x_1 and x_3.

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